

## SOME LOGICALLY WEAK RAMSEYAN THEOREMS

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**ABSTRACT.** We study four families of consequences of Ramsey's Theorems from a viewpoint of reverse mathematics. The first, what we call Achromatic Ramsey Theorems, is from a partition relation introduced by Erdős, Hajnal and Rado:  $\omega \rightarrow [\omega]_{c, \leq d}^r$ , which asserts that for every  $f : [\omega]^r \rightarrow c$  there exists an infinite  $H$  with  $|f([H]^r)| \leq d$ . The second and third are Free Set Theorems and Thin Set Theorems, which are introduced by Harvey Friedman. And the last is Rainbow Ramsey Theorems. We show that, most theorems from these families are quite weak, i.e., they are strictly weaker than  $\text{ACA}_0$ . Interestingly, these families turn out to be closely related. We establish the weakness of Achromatic Ramsey Theorems by an induction of exponents, then use this weakness and a similar induction to obtain weakness of Free Set Theorems, and derive weakness of Thin Set Theorems and Rainbow Ramsey Theorems as consequences.

## 1. INTRODUCTION

Reverse mathematics of Ramsey theory has been an active subject for computability theorists for years, in which Ramsey's Theorem for pairs ( $\text{RT}_2^2$ ) has enjoyed being the focus, perhaps since the work of Jockusch [11]. To facilitate following discussions of Ramsey theory, let us recall some terms. If  $X$  is a set and  $0 < r < \omega$ , then  $[X]^r$  is the set of  $r$ -element subsets of  $X$ ; when we write  $[X]^r$ ,  $r$  is always a positive integer. A function  $f$  is also called a *coloring* or a *partition*, and its values are natural numbers called *colors*. A *finite coloring* or  *$c$ -coloring* is a function with finite range or with range contained by  $c = \{0, 1, \dots, c-1\}$  where  $c$  is a positive integer. For a finite coloring  $f : [\omega]^r \rightarrow c$ , a set  $H$  is *homogeneous for  $f$*  if  $f$  is constant on  $[H]^r$ .

**Ramsey's Theorem.** *Every  $f : [\omega]^r \rightarrow c$  where  $c$  and  $r$  are positive integers, admits an infinite homogeneous set.*

For fixed  $r$  and  $c$ ,  $\text{RT}_c^r$  is the instance of Ramsey's Theorem for all  $f : [\omega]^r \rightarrow c$ .

In [11], Jockusch conjectured that computable two colorings of pairs may have all infinite homogeneous sets computing the halting problem. Speaking in reverse mathematics, we may formulate Jockusch's conjecture as:  $\text{RCA}_0 + \text{RT}_2^2 \vdash \text{ACA}_0$ . This conjecture was later refuted by Seetapun [18]. In his ingenious proof, Seetapun exploited the power of  $\Pi_1^0$  classes in controlling complexity, which is encapsulated in a theorem of Jockusch and Soare [12]. Seetapun's proof was later analyzed by Cholak, Jockusch and Slaman [2]. In [18, 2], several questions were raised: whether  $\text{RT}_2^2$  implies  $\text{WKL}_0$ ; whether Ramsey's Theorem for stable 2-colorings of pairs ( $\text{SRT}_2^2$ ) is equivalent to  $\text{RT}_2^2$ ; and whether  $\text{RT}_2^2$  implies  $\text{I}\Sigma_2$ . Of course, all these questions are based on  $\text{RCA}_0$ , which is a base theory for most works in

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reverse mathematics. These had been major open questions in reverse mathematics of Ramsey theory. The first two have been negatively answered by Jiayi Liu [16] and Chong, Slaman and Yang in [3] respectively. More recently, Chong, Slaman and Yang announced a negative answer to the third question.

Besides these major open questions, people have also studied consequences of Ramsey's Theorems, mostly of  $RT_2^2$ . Many consequences of  $RT_2^2$  have been shown to be strict weaker; and relations between these consequences give rise to a complicated picture, which fits the tradition of computability theory quite well. For example, Hirschfeldt and Shore [10] proved that the Ascending and Descending Sequences principle (ADS) and the Chain and Antichain principle (CAC) are both strictly weaker than  $RT_2^2$ ; Csima and Mileti [5] proved that Rainbow Ramsey Theorem for pairs ( $RRT_2^2$ ) does not imply ADS, and thus is also strictly weaker than  $RT_2^2$ .

Now we turn to Ramsey theory of general exponents, which seems desiring more attention from computability theorists. But we do know something, in particular, about complexity bounds. The pioneering work of Jockusch [11] gave some influential answers.

**Theorem 1.1** (Jockusch [11]). *Every computable finite coloring of  $[\omega]^r$  admits an infinite homogeneous set in  $\Pi_r^0$ . On the other hand, for each  $r > 1$ , there are computable 2-colorings of  $[\omega]^r$  which admit no infinite homogeneous sets in  $\Sigma_r^0$ .*

People have found that the complexity bounds above also appear in various consequences of  $RT_2^r$ . Cholak, Guisto, Hirst and Jockusch [1] showed that the  $\Pi_r^0/\Sigma_r^0$  bounds apply for Free Set Theorems and Thin Set Theorems; by Csima and Mileti [5], these bounds apply for Rainbow Ramsey Theorems; and by Chubb, Hirst and McNicholl [4], same bounds apply for a binary tree version of Ramsey's Theorems.

Concerning provability, we also have a few results. By Jockusch [11], we learn that  $RT_2^3$  is equivalent to  $ACA_0$  over  $RCA_0$ . Recently, the author has proved that  $RCA_0 + RRT_2^3 \not\vdash ACA_0$  in [21]; and later in [20] obtained some strengthening that  $RCA_0 + RRT_2^3 \not\vdash WKL_0$  and  $RCA_0 + RRT_2^3 \not\vdash RRT_2^4$ .

The aim of this paper is to study Ramsey theory of larger exponents, mainly from provability viewpoint. We consider several families of consequences of Ramsey's Theorems. As usual, we take  $RCA_0$  as the base theory and may assume it without explicit reference.

The first family is introduced by Erdős, Hajnal and Rado [8]:

$$\omega \rightarrow [\omega]_{c,<d}^r$$

if and only if for every  $c$ -coloring  $f$  of  $[\omega]^r$ , there exists an infinite  $H$  such that  $|f([H]^r)| < d$ . We call these partition relations *Achromatic Ramsey Theorems* (ART), and write  $ART_{c,d}^r$  for  $\omega \rightarrow [\omega]_{c,<d+1}^r$  and  $ART_{<\infty,d}^r$  for  $\forall c \text{ } ART_{c,d}^r$ .

The second and third families are *Free Set Theorems* and *Thin Set Theorems*, which are introduced by Harvey Friedman when he developed Boolean Relation Theory (see [9]). For a function  $f : [\omega]^r \rightarrow \omega$ , a set  $H$  is *free* if  $f(x_0, \dots, x_{r-1}) \notin H - \{x_0, \dots, x_{r-1}\}$  for all  $(x_0, \dots, x_{r-1}) \in [H]^r$ ; and a set  $S$  is *thin* for  $f$  if  $f([S]^r) \neq \omega$ . Free Set Theorems (FS) assert that every  $f : [\omega]^r \rightarrow \omega$  admits an infinite free set, and Thin Set Theorems (TS) assert that every  $f$  as above admits an infinite thin set.  $FS^r$  and  $TS^r$  are instances of FS and TS for fixed exponent  $r$  respectively. Cholak et al. [1] proved that  $RT_2^r$  implies  $FS^r$  and  $FS^r$  implies  $TS^r$ .

The last family is *Rainbow Ramsey Theorems*. Rainbow Ramsey Theorems concern bounded colorings: a coloring  $f : [\omega]^r \rightarrow \omega$  is *b-bounded*, if  $|f^{-1}(c)| \leq b$  for all  $c$ . A *rainbow* for a coloring  $f : [\omega]^r \rightarrow \omega$  is a set  $H$  such that  $f$  is injective on  $[H]^r$ . Rainbow Ramsey Theorems (RRT) assert that for every pair of positive integers  $b$  and  $r$ , every  $b$ -bounded coloring of  $[\omega]^r$  admits an infinite rainbow, and  $RRT_b^r$  is the

instance of Rainbow Ramsey Theorem for fixed exponent  $r$  and bound  $b$ . Galvin gave an easy proof of  $\text{RRT}_2^r$  from  $\text{RT}_2^r$  (see [5]), which can be easily translated to yield  $\text{RCA}_0 \vdash \forall r(\text{RT}_2^r \rightarrow \text{RRT}_2^r)$ .

It turns out that these families are closely related and share same weakness. We show that, for positive integers  $r$  and sufficiently large  $d$ , neither  $\text{ART}_{<\infty,d}^r$  nor  $\text{FS}^r$  implies  $\text{ACA}_0$ . Thus  $\text{TS} \not\vdash \text{ACA}_0$ . Moreover, we show that  $\text{FS}^r \vdash \text{RRT}_2^r$  and consequently  $\text{RRT} \not\vdash \text{ACA}_0$ . So, we negatively answer Question 7.6 in [1] and Question 5.15 in [5].

As one can predict, we establish weakness of Achromatic Ramsey Theorems and Free Set Theorems by proving some cone avoiding theorems and then building Turing ideals which do not contain the halting problem. From a viewpoint of model theory, it may be slightly more natural to read this common method as building a model which omits certain second order types.

**Definition 1.2.** Suppose that  $\mathcal{C}$  is a subset of reals, and  $\Phi = \forall X \exists Y \varphi(X, Y)$  is a  $\Pi_2^1$  sentence, where  $\varphi$  is an arithmetic formula with second order parameters.

- (1) For a fixed  $X$ , a set  $Y$  with  $\varphi(X, Y)$  is called a *solution* (of  $\Phi$  with respect to  $X$ ).
- (2) If for every  $X$  which computes no real in  $\mathcal{C}$ , there exists a solution  $Y$  such that  $X \oplus Y$  computes no real in  $\mathcal{C}$  either, then we say that  $\Phi$  admits  $\mathcal{C}$ -*omitting*.
- (3) If for any  $X$  (in  $\mathcal{C}$  or not) there exists a solution  $Y$  such that  $Y$  computes no real in  $\mathcal{C}$ , then we say that  $\Phi$  admits *strong  $\mathcal{C}$ -omitting*.
- (4) Suppose that  $\Phi$  admits (strong)  $\mathcal{C}$ -omitting for all  $A \not\leq_T B$  and  $\mathcal{C} = \{Z : A \leq_T B \oplus Z\}$ , then  $\Phi$  has (*strong*) *cone avoidance property*.

Suppose that  $p$  is a second order type which can not be satisfied by computable reals. If  $\Phi$  admits omitting for the set of reals satisfying  $p$ , then we can build a countable  $\omega$ -model of  $\text{RCA}_0 + \Phi$  omitting  $p$ . So, cone avoidance property is sufficient for proving  $\Phi \not\vdash \text{ACA}_0$ . But, most theorems in the four families enjoy strong cone avoidance property. To be precise, for each  $r$ ,  $\text{ART}_{<\infty,d}^r$  has strong cone avoidance property for sufficiently large  $d$ ; and  $\text{FS}^r$ ,  $\text{TS}^r$  and  $\text{RRT}_2^r$  all have strong cone avoidance property. Actually, strong cone avoidance property is a key factor, which allows us to establish cone avoidance by inductions on exponents for Achromatic Ramsey Theorems and Free Set Theorems. Interestingly, these inductions follow a zig-zag pattern. For example, the induction for FS goes like: from induction hypothesis that strong cone avoidance property holds for  $\text{FS}^{<r}$ , we obtain cone avoidance property for  $\text{FS}^r$ ; and apply this weaker property to get strong cone avoidance property for  $\text{FS}^r$ .

The proofs of strong cone avoidance property for Achromatic Ramsey Theorems and Free Set Theorems share some other common features. Both proofs follow the analysis of Seetapun's theorem by Cholak, Jockusch and Slaman [2]: we obtain some cone avoiding set which has some property like cohesiveness; then build a desired set as a subset of this cohesive-like set. And in both proofs, we use Mathias forcing and exploit the power of  $\Pi_1^0$  classes in controlling complexity of certain Mathias generics, as Seetapun did in his celebrated proof.

Besides these similarities, the proof of strong cone avoidance for Free Set Theorems heavily depends of strong cone avoidance property of Achromatic Ramsey Theorems. So, all clues here suggest that there are some deeper relations, perhaps metamathematical relations, between Achromatic Ramsey Theorems and Free Set Theorems. However, at this moment we know nothing. We put relating questions in the last section.

Below, we briefly introduce the remaining sections:

- In §2, we introduce some conventions to facilitate technical formulations, and also some basic properties of Mathias forcing.
- In §3, we establish strong cone avoidance property of Achromatic Ramsey Theorems and Thin Set Theorems, and also weakness of Achromatic Ramsey Theorems and Thin Set Theorems.
- In §4, we establish strong cone avoidance property and weakness of Free Set Theorems. In addition, we reduce Rainbow Ramsey Theorems to Free Set Theorems, and thus obtain strong cone avoidance property of RRT.
- In §5, we raise some questions.

## 2. PRELIMINARIES

In this section, we setup some conventions and recall some notions and known results which are useful for our purposes. For more background knowledge in computability and reverse mathematics, we refer readers to [15] and [19]. We also need some elementary algorithmic randomness, which could be found in [17].

**2.1. Sequences.** If  $s$  and  $t$  are two finite sequences, then we write  $st$  for the concatenation of  $s$  and  $t$ . If  $x$  is a single symbol, then  $\langle x \rangle$  is the finite sequence with only one symbol  $x$ . The length of a finite sequence  $s$  is denoted by  $|s|$ . If  $l < |s|$  then  $s \upharpoonright l$  is the initial segment of  $s$  of length  $l$ . For  $X \subseteq \omega$ ,  $X \upharpoonright l$  is interpreted as an initial segment of the characteristic function of  $X$  in the obvious way.

Recall that  $[X]^r$  for  $0 < r < \omega$  is the set of  $r$ -element subsets of  $X$ . We also write  $[X]^\omega$  for the set of countable subsets of  $X$ ;  $[X]^{<r}$ ,  $[X]^{\leq r}$ ,  $[X]^{<\omega}$ ,  $[X]^{\leq \omega}$  are interpreted naturally. If  $X \subseteq \omega$ , then elements of  $[X]^{\leq \omega}$  are identified with strictly increasing sequences. We use  $\sigma, \tau, \dots$  for elements of  $[\omega]^{<\omega}$ . Under the above convention, we may perform both sequence operations and set operations on elements of  $[\omega]^{<\omega}$ . For example, we can write  $\sigma\tau$  for  $\sigma \cup \tau$ , if  $\max \sigma < \min \tau$ ;  $\sigma \subseteq \tau$  if  $\sigma$  is a subset of  $\tau$ ; and  $\sigma - \tau = \{x \in \sigma : x \notin \tau\}$ . We extend this convention to infinite subsets of  $\omega$ , so we write  $\sigma X$  for  $\sigma \cup X$ , if  $\max \sigma < \min X$  and  $X \in [\omega]^{\leq \omega}$ .

**2.2. Trees.** We work with trees which are subsets of  $\omega^{<\omega}$ . If  $T$  is a finite tree, then

$$[T] = \{\sigma \in T : \forall x(\sigma \langle x \rangle \notin T)\};$$

if  $T$  is an infinite tree, then  $[T]$  denote the set of infinite sequences whose initial segments are always in  $T$ .

When we use finite trees for measure theoretic arguments, we define a function  $m_T$  for each finite tree  $T$  by induction:  $m_T(\emptyset) = 1$ , if  $\sigma \langle x \rangle \in T$  then

$$m_T(\sigma \langle x \rangle) = \frac{m_T(\sigma)}{|\{y : \sigma \langle y \rangle \in T\}|}.$$

We should consider  $m_T$  as a probability measure associated to  $T$ . So, we can naturally extend the domain of  $m_T$  to include certain subsets of  $T$ , and denote the resulting function by  $m_T$  too: if  $S \subseteq T$  is *prefix-free* (i.e., if  $S$  contains  $\sigma$  then  $S$  contains *no* proper initial segment of  $\sigma$ ), typically  $S \subseteq [T]$ , then

$$m_T S = \sum_{\sigma \in S} m_T(\sigma).$$

**2.3. Computations.** For a finite sequence  $\sigma$ , we write  $\Phi_e(\sigma; x) \downarrow$  if  $\Phi_e(\sigma; x)$  converges in  $|\sigma|$  many steps. We write  $\Phi_e(\sigma; x) \uparrow$  for  $\neg(\Phi_e(\sigma; x) \downarrow)$ .

For a set  $B$  and a finite sequence  $\sigma$ , we write  $\Phi_e^B(\sigma; x) \downarrow$  if  $\Phi_e((B \upharpoonright |\sigma|) \oplus \sigma; x) \downarrow$ . Notations, like  $\Phi_e^B(\sigma; x) \uparrow$  and  $\Phi_e^B(X)$ , are interpreted in similar way.

To force a non-computability statement like  $\Phi_e^B(H) \neq A$ , splitting computations are usually helpful. A pair  $(\eta_0, \eta_1) \in [\omega]^{<\omega} \times [\omega]^{<\omega}$  is  $(e, B)$ -*splitting* over  $\sigma \in [\omega]^{<\omega}$ , if  $\max \sigma < \min \eta_i$  for  $i < 2$  and  $\Phi_e^B(\sigma \eta_0; x) \downarrow \neq \Phi_e^B(\sigma \eta_1; x) \downarrow$  for some  $x$ .

**2.4. Some useful known results.** We list some useful results here, but formulate some of them in terms of Definition 1.2

**Theorem 2.1** (Jockusch and Soare [12]).  $\text{WKL}_0$  has cone avoidance property.

Theorem 2.1 reflects the power of  $\Pi_1^0$  classes in controlling complexity, and plays an important role in Seetapun's proof of the following theorem.

**Theorem 2.2** (Seetapun [18]).  $\text{RT}_2^2$  has cone avoidance property

Dzhafarov and Jockusch discovered a neglected feature of Seetapun's proof that the proof works for finite partitions of  $\omega$  of arbitrary complexity.

**Theorem 2.3** (Dzhafarov and Jockusch [7]). *Infinite pigeonhole principle has strong cone avoidance property.*

For Free Set Theorems, we need the following theorem.

**Theorem 2.4** (Cholak et al. [1]). *For each  $f : [\omega]^r \rightarrow \omega$ , there exists  $g : [\omega]^r \rightarrow 2r + 2$  such that  $g \leq_T f$  and  $g \oplus H$  computes an infinite  $f$ -free set for every infinite  $g$ -homogeneous  $H$ . Moreover, if  $f(\sigma) \leq \max \sigma$  for all  $\sigma \in [\omega]^r$  then every  $g$ -homogeneous set is  $f$ -free.*

Note that, combining Theorems 2.4 and 2.3, if we restrict Free Set Theorems for  $f : \omega \rightarrow \omega$  such that  $f(x) \leq x$  for all  $x \in \omega$ , then we have strong cone avoidance property.

**2.5. Mathias forcing.** Here we include some well-known computability theoretic property of Mathias forcing and also an easy corollary of this property that COH has strong cone avoidance property.

**Definition 2.5.** A *Mathias condition* is a pair  $(\sigma, X) \in [\omega]^{<\omega} \times [\omega]^\omega$  such that  $\max \sigma < \min X$ . We identify a Mathias condition  $(\sigma, X)$  with the set below:

$$\{Y \in [\omega]^\omega : \sigma \subset Y \subseteq \sigma \cup X\}.$$

For two Mathias conditions  $(\sigma, X)$  and  $(\tau, Y)$ ,  $(\tau, Y) \leq_M (\sigma, X)$  if and only if  $(\tau, Y) \subseteq (\sigma, X)$  under the above convention.

**Lemma 2.6** (Logicians). *For each  $e$  and a Mathias condition  $(\sigma, X)$  with  $A \not\leq_T B \oplus X$ , there exists a Mathias condition  $(\tau, Y) \leq_M (\sigma, X)$ , such that  $A \not\leq_T B \oplus Y$  and  $\Phi_e^B(Z) \neq A$  for every  $Z \in (\tau, Y)$ .*

*Proof.* There are two cases.

*Case 1:*  $X$  contains a pair  $(\eta_0, \eta_1)$  which  $(e, B)$ -spits over  $\sigma$ .

Fix  $i < 2$  and  $x$  such that  $\Phi_e^B(\sigma\eta_i; x) \downarrow \neq A(x)$ . Let  $\tau = \sigma\eta_i$  and  $Y = X \cap (\max \eta_i, \infty)$ . Then  $(\tau, Y)$  is as desired.

*Case 2:*  $X$  contains no pair  $(e, B)$ -splitting over  $\sigma$ .

If  $Z \in [X]^\omega$  and  $\Phi^B(Z)$  is total then  $\Phi^B(Z) \leq_T B \oplus X$ , and thus  $\Phi^B(Z) \neq A$ , since  $A \not\leq_T B \oplus X$ . So we can simply let  $(\tau, Y) = (\sigma, X)$ .  $\square$

The following theorem is an easy corollary of the above lemma and Theorem 2.3. Recall that an infinite set  $C$  is *cohesive* for a sequence  $\vec{R} = (R_n : n < \omega)$ , if and only if for each  $n$  either  $C \cap R_n$  or  $C - R_n$  is finite. COH, a consequence of  $\text{RT}_2^2$  introduced by Cholak, Jockusch and Slaman [2], asserts that every sequence admits a cohesive set.

**Theorem 2.7** (Logicians). *COH has strong cone avoidance property.*

## 3. ACHROMATIC RAMSEY THEOREMS

In this section, we prove that  $\text{ART}_{<\infty,d}^r$  has strong cone avoidance property for appropriate  $d$ .

**Theorem 3.1.** *For each  $r > 0$ , there exists  $d$  such that  $\text{ART}_{<\infty,d}^r$  has strong cone avoidance property. Hence,  $\text{ART}_{<\infty,d}^r \not\models \text{ACA}_0$  for sufficiently large  $d$ .*

Clearly, the second part of Theorem 3.1 is a consequence of the first part. Before proving the first part of Theorem 3.1, we present some easy corollaries.

**Theorem 3.2.** *TS has strong cone avoidance property. Thus  $\text{TS} \not\models \text{ACA}_0$ .*

*Proof.* Fix  $X \not\leq_T Y$  and  $f : [\omega]^r \rightarrow \omega$  with  $r > 0$ . By strong cone avoidance of Achromatic Ramsey Theorems, let  $d$  be such that  $\text{ART}_{d+1,d}^r$  has strong cone avoidance property. For each  $\sigma \in [\omega]^r$ , let  $g(\sigma) = \min\{d, f(\sigma)\}$ . So,  $g : [\omega]^r \rightarrow d+1$ . Pick  $Z \in [\omega]^\omega$  such that  $X \not\leq Y \oplus Z$  and  $|g([Z]^r)| \leq d$ . Then  $Z$  is clearly thin for  $f$ . So, TS has strong cone avoidance property.

Hence,  $\text{TS} \not\models \text{ACA}_0$ .  $\square$

As many other consequences of Ramsey's Theorems, Achromatic Ramsey Theorems also obey the bounds of Jockusch in Theorem 1.1.

**Proposition 3.3.** *Fix  $r \geq 2$ ,  $c \geq 2$  and  $d > 0$ .*

- (1) *For each computable  $f : [\omega]^r \rightarrow c$ , there exists an infinite  $H \in \Pi_r^0$  such that  $|f([H]^r)| \leq d$ .*
- (2) *There exists a computable  $g : [\omega]^r \rightarrow d+1$ , such that no infinite  $H \in \Sigma_r^0$  can have  $|f([H]^r)| \leq d$ .*

*Proof.* (1) follows easily from Theorem 1.1.

On the other hand, Cholak et al. [1] define a computable  $h : [\omega]^r \rightarrow \omega$  which admits no infinite thin sets in  $\Sigma_r^0$ . So (2) follows from this known bound and the proof of Theorem 3.2.  $\square$

**Corollary 3.4.** *For  $r > 2$  and  $c > d > 0$ ,  $\text{RT}_2^2 \not\models \text{ART}_{c,d}^r$ . Consequently,  $\text{ART}_{4,3}^3$  is strictly between  $\text{RT}_2^2$  and  $\text{RT}_2^3$  and  $\text{ART}_{3,2}^3$  is strictly between  $\text{RT}_{<\infty}^2$  and  $\text{RT}_2^3$ , where  $\text{RT}_{<\infty}^2$  is  $\forall n \text{RT}_n^2$ .*

*Proof.* By relativizing Theorem 3.1 of Cholak, Jockusch and Slaman [2], there exists an  $\omega$ -model  $\mathcal{M}$  of  $\text{RCA}_0 + \text{RT}_2^2$  containing only  $\Delta_3^0$  sets. By Proposition 3.3(2),  $\mathcal{M} \not\models \text{ART}_{c,d}^r$ .

The above  $\omega$ -model is also a model of  $\text{RT}_{<\infty}^2$ . On the other hand, Dorais et al. [6, §5] prove that  $\text{ART}_{4,3}^3 \vdash \text{RT}_2^2$  and  $\text{ART}_{3,2}^3 \vdash \text{RT}_{<\infty}^2$ .  $\square$

We prove the first part of Theorem 3.1 by induction on  $r$ , that  $\text{ART}_{<\infty,d}^r$  has strong cone avoidance property for sufficiently large  $d$ . The induction goes in a zig-zag way:

- (A1) As infinite pigeonhole principle has strong cone avoidance property, we get strong cone avoidance property of  $\text{ART}_{<\infty,1}^1$ .
- (A2) Fix  $(d_k : 0 < k < r)$ , such that  $\text{ART}_{<\infty,d_k}^k$  has strong cone avoidance property, for each  $k \in (0, r)$ . Firstly we prove that  $\text{ART}_{<\infty,d_{r-1}}^r$  has cone avoidance property.
- (A3) Then we prove that  $\text{ART}_{<\infty,d}^r$  has strong cone avoidance property for

$$d = d_{r-1} + \sum_{0 < k < r} d_k d_{r-k}.$$

(A1) is done. (A2) is accomplished by the lemma below.

**Lemma 3.5.** *If  $\text{ART}_{c,e}^n$  has strong cone avoidance property, then  $\text{ART}_{c,e}^{n+1}$  has cone avoidance property.*

*Proof.* Fix  $X, Y$  and  $g : [\omega]^{n+1} \rightarrow c$  such that  $X \not\leq_T Y \oplus g$ .

For each  $\sigma \in [\omega]^n$  and  $k < c$ , let  $R_{\sigma,k} = \{x : g(\sigma \langle x \rangle) = k\}$ . By cone avoidance property of COH, pick  $Z$  such that  $X \not\leq_T Y \oplus g \oplus Z$  and  $Z$  is cohesive for  $(R_{\sigma,k} : \sigma \in [\omega]^n, k < c)$ . For each  $\sigma \in [\omega]^n$ , let  $\bar{g}(\sigma) = \lim_{x \in Z} g(\sigma \langle x \rangle)$ , which is defined by cohesiveness of  $Z$ . By strong cone avoidance of  $\text{ART}_{c,e}^n$ , pick  $W \in [Z]^\omega$  such that  $X \not\leq_T Y \oplus g \oplus W$  and  $|\bar{g}([W]^n)| \leq e$ . Let  $\theta = \bar{g}([W]^n)$ .

We build a strictly increasing sequence  $(\sigma_s \in [W]^{<\omega} : s < \omega)$  by induction. Let  $\sigma_0 = \emptyset$ . Suppose that  $\sigma_s \in [W]^{<\omega}$  and  $g([\sigma_s]^{n+1}) \subseteq \theta$ . As  $\bar{g}([\sigma_s]^n) \subseteq \theta$ ,  $g(\rho \langle x \rangle) = \bar{g}(\rho) = \lim_{s \in W} g(\rho \langle s \rangle) \in \theta$  for all  $\rho \in [\sigma_s]^n$  and sufficiently large  $x \in W$ . So, in a  $g \oplus W$ -computable way, we can pick

$$x_s = \min\{x \in W : x > \max \sigma_s \wedge \forall \rho \in [\sigma_s]^n (g(\rho \langle x \rangle) \in \theta)\}.$$

Let  $\sigma_{s+1} = \sigma_s \langle x_s \rangle$ .

So,  $V = \bigcup_s \sigma_s$  is  $g \oplus W$ -computable and infinite, and  $g([V]^{n+1}) \subseteq \theta$ . Moreover,  $X \not\leq_T Y \oplus g \oplus V$ , as  $X \not\leq_T Y \oplus g \oplus W$ .  $\square$

The remaining part of this section is devoted to (A3). Fix  $A \not\leq_T B$  and  $f : [\omega]^r \rightarrow c$  where  $c < \omega$ . For each  $k < r$  and  $\rho \in [\omega]^k$ , let  $f_\rho(\tau) = f(\rho \tau)$  for  $\tau \in [\omega]^{r-k}$  with  $\max \rho < \min \tau$ . In this section, we say that a set  $X$  is *cone avoiding* if  $A \not\leq_T B \oplus X$ ; and a Mathias condition  $(\sigma, X)$  is *cone avoiding*, if  $X$  is cone avoiding.

We need a cone avoiding infinite  $H$  such that  $|f([H]^r)| \leq d$ . We build such an  $H$  in several steps:

- (1) By the induction hypothesis and Mathias forcing, we build a cone avoiding  $D \in [\omega]^\omega$  and a sequence  $(\Theta_k : 0 < k < r)$ , such that
  - (a) each  $\Theta_k$  is a set of at most  $d_k$  many sets of colors, and  $|\theta| \leq d_{r-k}$  for each  $\theta \in \Theta_k$ ;
  - (b) if  $0 < k < r$  and  $\rho \in [D]^k$ , then there exist  $\theta_\rho \in \Theta_k$  and  $b$ , so that  $f_\rho(\tau) \in \theta_\rho$  for all  $\tau \in [D \cap (b, \infty)]^{r-k}$ .
- (2) By a Seetapun-style Mathias forcing, we build a cone avoiding  $G \in [D]^\omega$  as the union of some  $(\xi_n \in [D]^{<\omega} : n < \omega)$ , such that
  - (a)  $|f([\xi_n]^r)| \leq d_{r-1}$  and  $\max \xi_n < \min \xi_{n+1}$ ;
  - (b)  $f_\rho(\tau) \in \theta_\rho \in \Theta_k$ , for  $k \in (0, r)$ ,  $\rho \in [\bigcup_{i < n} \xi_i]^k$  and  $\tau \in [\bigcup_{i \geq n} \xi_i]^{r-k}$ .
- (3) By strong cone avoidance of infinite pigeonhole principle, we build  $H$  as a subset of  $G$ .

**3.1. The construction of  $D$ .** Firstly, we build a cone avoiding  $C \in [\omega]^\omega$  and a sequence  $(\theta_\rho : 0 < |\rho| < r)$ , such that for each  $k \in (0, r)$  and  $\rho \in [\omega]^k$ ,

- (C1)  $\theta_\rho$  is a subset of  $c$  with at most  $d_{r-k}$  many elements;
- (C2)  $f_\rho(\tau) \in \theta_\rho$  for all  $\tau \in [C]^{r-k}$  with  $\min \tau$  sufficiently large.

Note that, (C2) implies that  $C$  has some kind of cohesiveness. Thus, it is not surprising that the construction of  $C$  looks like a construction of cohesive sets.

**Lemma 3.6.** *Suppose that  $0 < k < r$  and  $\rho \in [\omega]^k$ . Then every cone avoiding Mathias condition  $(\sigma, X)$  can be extended to another cone avoiding  $(\sigma, Y)$  such that  $\max \rho < \min Y$  and  $|f_\rho([Y]^{r-k})| \leq d_{r-k}$ .*

*Proof.* As  $\text{ART}_{c,d_{r-k}}^{r-k}$  has strong cone avoidance property, we can pick a cone avoiding  $Y \in [X]^\omega$  such that  $\max \rho < \min Y$  and  $|f_\rho([Y]^{r-k})| \leq d_{r-k}$ .  $\square$

With the above lemma and Lemma 2.6, we can obtain a descending sequence of cone avoiding Mathias conditions  $((\sigma_n, X_n) : n < \omega)$  and a sequence  $(\theta_\rho : 0 < |\rho| < r)$ , satisfying the following properties:

- (1) If  $k \in (0, r)$  and  $\rho \in [\omega]^k$ , then  $\theta_\rho \in [c]^{\leq d_{r-k}}$  and there exist  $n$  and such that  $f_\rho([X_n]^{r-k}) = \theta_\rho$ ;
- (2) For each  $n$ ,  $|\sigma_n| < |\sigma_{n+1}|$  and  $\Phi_n^B(Z) \neq A$  for all  $Z \in (\sigma_n, X_n)$ .

So (C1) and (C2) hold for  $C = \bigcup_n \sigma_n$  and  $(\theta_\rho : 0 < |\rho| < r)$ , and  $C$  is cone avoiding.

Secondly, we build a desired  $D \in [C]^\omega$ . For each  $k \in (0, r)$ , define  $F_k(\rho) = \theta_\rho$  for  $\rho \in [C]^k$ . As  $\text{ART}_{<\infty, d_k}^k$  has strong cone avoidance property for each  $k \in (0, r)$ , we can obtain a sequence  $(D_k : k < r)$ , such that

- (1)  $D_0 = C$  and  $D_{k+1} \in [D_k]^\omega$  is cone avoiding for each  $k < r - 1$ .
- (2)  $|F_k([D_k]^k)| \leq d_k$  if  $0 < k < r$ .

Let  $D = D_{r-1}$ . For each  $k \in (0, r)$ , let  $\Theta_k = F_k([D]^k)$ . It follows that

- (D) if  $0 < k < r$  and  $\rho \in [D]^k$ , then  $f(\rho\tau) \in \theta_\rho \in \Theta_k$  for all  $\tau \in [D]^{r-k}$  with  $\min \tau$  sufficiently large.

**3.2. The construction of  $G$ .** In this subsection, we start with  $D$  from §3.1 and build a sequence  $(\xi_n \in [D]^{<\omega} : n < \omega)$ , such that

- (G1)  $G = \bigcup_n \xi_n$  is infinite and cone avoiding;
- (G2)  $|f([\xi_n]^r)| \leq d_{r-1}$  and  $\max \xi_n < \min \xi_{n+1}$ ;
- (G3) For each  $k \in (0, r)$  and  $\rho \in [\bigcup_{i < n} \xi_i]^k$ ,  $f_\rho(\tau) = f(\rho\tau) \in \theta_\rho$  for all  $\tau \in [\bigcup_{i \geq n} \xi_i]^{r-k}$ .

Note that, if we ignore (G1) then we can easily get some  $(\eta_n \in [D]^{<\omega} : n < \omega)$  satisfying (G2) and (G3) in places of  $(\xi_n : n < \omega)$ . We start with  $(\sigma_0, X_0) = (\emptyset, D)$ , and extend  $(\sigma_n, X_n)$  to  $(\sigma_n, Y_{n+1})$ , so that  $f_\rho(\tau) = f(\rho\tau) \in \theta_\rho$  for  $k \in (0, r)$ ,  $\rho \in [\sigma_n]^k$  and  $\tau \in [Y_{n+1}]^{r-k}$ , then we extend  $(\sigma_n, Y_{n+1})$  to  $(\sigma_{n+1}, X_{n+1})$  with  $\sigma_{n+1} = \sigma_n \eta_n$  for some  $\eta_n$  of length 1. By (D), we can even make  $X_{n+1} = X_n \cap (b, \infty)$  for some  $b$ . However, in general we need  $(f \oplus D)'$  to find such a lower bound  $b$ , thus we can not ensure that  $\bigcup_n \eta_n$  is cone avoiding. So, the non-trivial job is to satisfy (G2, G3) and (G1) simultaneously. To this end, we follow Seetapun's celebrated proof in [18].

Let  $\mathcal{C}$  be the set of all  $c$ -colorings of  $[\omega]^r$ . Then  $f \in \mathcal{C}$  and  $\mathcal{C}$  is a  $\Pi_1^0$  class.

**Lemma 3.7.** *For each  $e$  and a cone avoiding Mathias condition  $(\sigma, X)$ , there exists a cone avoiding  $(\sigma\xi, Y) \leq_M (\sigma, X)$  such that  $|f([\xi]^r)| \leq d_{r-1}$  and  $\Phi_e^B(Z) \neq A$  for all  $Z \in (\sigma\xi, Y)$ .*

*Proof.* Let  $\mathcal{U}$  be the set of  $g \in \mathcal{C}$ , such that if  $\tau \in [X]^{<\omega}$  and  $|g([\tau]^r)| \leq d_{r-1}$  then  $\tau$  contains no pair  $(e, B)$ -splitting over  $\sigma$ . So,  $\mathcal{U}$  is  $\Pi_1^0$  in  $B \oplus X$ .

*Case 1:  $\mathcal{U} = \emptyset$ .* In particular,  $f \notin \mathcal{U}$ .

By the definition of  $\mathcal{U}$ , we can pick  $\xi_0$  and  $\xi_1$  from  $[X]^{<\omega}$  and  $x$ , so that  $|f([\xi_i]^r)| \leq d_{r-1}$  for  $i < 2$  and  $\Phi_e^B(\sigma\xi_0; x) \downarrow \neq \Phi_e^B(\sigma\xi_1; x) \downarrow$ . Fix  $i < 2$  such that  $\Phi_e^B(\sigma\xi_i; x) \neq A(x)$  and let  $\xi = \xi_i$ . So,  $(\sigma\xi, X \cap (\max \xi_i, \infty))$  is a cone avoiding extension as desired.

*Case 2:  $\mathcal{U} \neq \emptyset$ .*

As  $X$  is cone avoiding, by cone avoidance property of  $\text{WKL}_0$  (Theorem 2.1) there exists  $g \in \mathcal{U}$  with  $X \oplus g$  cone avoiding. By Lemma 3.5 and the induction hypothesis that  $\text{ART}_{c, d_{r-1}}^{r-1}$  has strong cone avoidance property, pick  $Y \in [X]^\omega$  such that  $Y$  is cone avoiding and  $|g([Y]^r)| \leq d_{r-1}$ . As  $g \in \mathcal{U}$ ,  $Y$  contains no pair  $(e, B)$ -splitting over  $\sigma$ . So, if  $Z \in (\sigma, Y)$  and  $\Phi_e^B(Z)$  is total then  $\Phi_e^B(Z) \leq_T B \oplus Y$  and thus  $\Phi_e^B(Z) \neq A$ . Thus,  $(\sigma, Y)$  is a desired extension.  $\square$

By the construction of  $C$ , every Mathias condition  $(\sigma, X)$  with  $X \subseteq C$  can be extended to some  $(\tau, Y) = (\sigma, X \cap (b, \infty))$ , such that  $f_\rho(v) \in \theta_\rho$  for all non-empty  $\rho \in [\tau]^{<r}$  and  $v \in [Y]^{r-|\rho|}$ .



By the above remark and Lemma 3.7, we can build a descending sequence of cone avoiding Mathias conditions  $((\sigma_n, X_n) : n < \omega)$ , such that

- (1)  $(\sigma_0, X_0) = (\emptyset, D)$ ;
- (2)  $f_\rho(\tau) \in \theta_\rho$  for all non-empty  $\rho \in [\sigma_n]^{<r}$  and  $\tau \in [X_n]^{r-|\rho|}$ ;
- (3)  $\sigma_{n+1} = \sigma_n \xi_n$  for some non-empty  $\xi_n$  with  $|f([\xi_n]^r)| \leq d_{r-1}$ ;
- (4)  $\Phi_n^B(Z) \neq A$  for all  $Z \in (\sigma_{n+1}, X_{n+1})$ .

Let  $G = \bigcup_n \xi_n = \bigcup_n \sigma_n$ . Then (G1-3) are satisfied.

**3.3. The construction of  $H$ .** For each  $n$ , let  $\alpha_n = f([\xi_n]^r)$ . Then  $\alpha_n$  is a subset of  $c$  with at most  $d_{r-1}$  many elements. For each  $\alpha \in [c]^{\leq d_{r-1}}$ , let

$$G_\alpha = \{x \in G : \exists n(x \in \xi_n \wedge \alpha_n = \alpha)\}.$$

By strong cone avoidance property of infinite pigeonhole principle (Theorem 2.3), there exist  $\alpha \in [c]^{\leq d_{r-1}}$  and a cone avoiding  $H \in [G_\alpha]^\omega$ .

**Lemma 3.8.**  $|f([H]^r)| \leq d$ .

*Proof.* Let  $\sigma = (x_0, \dots, x_{r-1}) \in [H]^r$  be arbitrary. If  $\sigma \subseteq \xi_n$  for some  $n$ , then  $f(\sigma) \in \alpha_n = \alpha$  by the definition of  $H$ . Suppose that  $x_{k-1} \in \xi_n$  and  $x_k > \max \xi_n$  for some  $k \in (0, r)$  and  $n$ . Let  $\rho = (x_0, \dots, x_{k-1})$  and  $\tau = (x_k, \dots, x_{r-1})$ . By (G3),  $f(\sigma) = f_\rho(\tau) \in \theta_\rho$ ; by (D),  $\theta_\rho \in \Theta_k$ . So,  $f(\sigma)$  is in

$$\alpha \cup \{i < c : \exists k \in (0, r), \theta \in \Theta_k(i \in \theta)\}.$$

So,  $|f([H]^r)| \leq d_{r-1} + \sum_{0 < k < r} d_k d_{r-k} = d$ . □

This completes the proof of Theorem 3.1.

#### 4. FREE SET THEOREMS

In this section, we establish the strong cone avoidance property for Free Set Theorems of arbitrary finite exponent.

**Theorem 4.1.** *FS has strong cone avoidance property. Hence,  $\text{FS} \not\vdash \text{ACA}_0$ .*

Before proving Theorem 4.1, we apply it to obtain similar results for Rainbow Ramsey Theorems.

**Theorem 4.2.** *For each  $n > 0$  and a 2-bounded function  $f$  on  $[\omega]^n$ , there exists a uniformly  $f$ -computable  $g : [\omega]^n \rightarrow \omega$  such that every  $g$ -free set is an  $f$ -rainbow.*

*Hence,  $\text{RRT}$  has strong cone avoidance property,  $\text{RCA}_0 \vdash \forall n > 0 (\text{FS}^n \rightarrow \text{RRT}_k^n)$  for every  $k < \omega$  and  $\text{RRT} \not\vdash \text{ACA}_0$ .*

*Proof.* By Theorem 4.1, it suffices to prove the first half.

Fix a computable bijection  $\ulcorner \cdot \urcorner : [\omega]^n \rightarrow \omega$ . Let  $f : [\omega]^n \rightarrow \omega$  be 2-bounded. For each  $\sigma \in [\omega]^n$ , let

$$g(\sigma) = \begin{cases} \min(\tau - \sigma), & \exists \tau (\ulcorner \tau \urcorner < \ulcorner \sigma \urcorner \wedge f(\sigma) = f(\tau)); \\ 0, & \text{otherwise.} \end{cases}$$

As  $f$  is 2-bounded, if  $\tau$  in the definition of  $g(\sigma)$  exists then it is unique. As  $\tau$  and  $\sigma$  are two distinct finite sets of same size,  $\tau - \sigma \neq \emptyset$ . Thus  $g$  is well defined and total. By  $\text{FS}^n$ , let  $X \in [\omega]^\omega$  be  $g$ -free.

We claim that  $X$  is a rainbow for  $f$ . Assume that  $f(\sigma) = f(\tau)$  for distinct  $\sigma, \tau \in [X]^n$ . Without loss of generality, assume that  $\ulcorner \tau \urcorner < \ulcorner \sigma \urcorner$ . Then,  $g(\sigma) \in \tau - \sigma \subset X - \sigma$ , and we have a desired contradiction. □

Below, we prove Theorem 4.1. Clearly, the second part is a consequence of the first part. To prove the first part, the overall plan is to establish strong cone avoidance property for  $\text{FS}^r$  by induction on the exponent  $r$ :

- (F1) Firstly we prove that  $\text{FS}^1$  has strong cone avoidance property;
- (F2) Then we establish cone avoidance property for  $\text{FS}^r$  with  $r > 1$ , with the induction hypothesis that  $\text{FS}^{r-1}$  has strong cone avoidance property;
- (F3) Finally we prove that  $\text{FS}^r$  has *strong* cone avoidance property for  $r > 1$ , with the full induction hypothesis for all lesser exponents.

A key idea to accomplish (F1) and (F3) is to reduce  $\text{FS}^r$  to  $\text{FS}^r$  for functions behaving tamely. We establish this reduction in Lemma 4.3 below.

For  $r > 0$ , each  $\sigma \in [\omega]^r$  induces a finite sequence of *traps* (i.e., intervals)  $(I_k^\sigma : k \leq r)$ , where

$$\begin{aligned} I_0^\sigma &= [0, \sigma(0)], \\ I_k^\sigma &= [\sigma(k-1), \sigma(k)] \text{ if } 0 < k < r, \\ I_r^\sigma &= (\sigma(r-1), \infty). \end{aligned}$$

For  $k \leq r$  and a function  $f : [\omega]^r \rightarrow \omega$ , we say that  $f$  is *k-trapped* if  $f(\sigma) \in I_k^\sigma$  for all  $\sigma \in [\omega]^r$ ; and  $f$  is *trapped* if it is *k-trapped* for some  $k$ .  $\text{FS}^r$  can be restricted to a certain class of functions, so we may say  $\text{FS}^r$  for *k-trapped functions*, etc.

**Lemma 4.3.** *If  $\text{FS}^r$  for trapped functions has (strong) cone avoidance property, then  $\text{FS}^r$  has (strong) cone avoidance property.*

*Proof.* We prove the lemma for strong cone avoidance property. The proof for cone avoidance is similar and thus omitted.

Fix  $A, B$  and  $f : [\omega]^r \rightarrow \omega$  such that  $A \not\leq_T B$ . For each  $\sigma \in [\omega]^r$ , let

$$\begin{aligned} f_0(\sigma) &= \min\{\sigma(0), f(\sigma)\}; \\ f_k(\sigma) &= \min\{\sigma(k), \max\{\sigma(k-1), f(\sigma)\}\} \text{ if } 0 < k < r; \\ f_r(\sigma) &= \max\{\sigma(r-1) + 1, f(\sigma)\}. \end{aligned}$$

By the assumption, we get  $(H_k : k \leq r)$  such that

- (1)  $H_0 \in [\omega]^\omega$  and  $H_k \in [H_{k-1}]^\omega$  if  $k > 0$ ;
- (2)  $A \not\leq_T B \oplus H_k$ ;
- (3)  $H_k$  is free for  $f_0, \dots, f_k$ .

We claim that  $H_r$  is free for  $f$ . Let  $\sigma \in [H_r]^r$  be arbitrary. Then  $f(\sigma) \in I_k^\sigma$  for some  $k \leq r$  and thus  $f(\sigma) = f_k(\sigma)$ . As  $H_r$  is free for  $f_k$ ,  $f(\sigma) \notin H_r - \sigma$ .  $\square$

So, it suffices to deal with  $\text{FS}^r$  for trapped functions. Among all trapped functions,  $r$ -trapped functions are the most easy going.

**Lemma 4.4.** *If  $f : [\omega]^r \rightarrow \omega$  is  $r$ -trapped and  $X$  is Martin-Löf random in  $f$ , then there exists an infinite  $X$ -computable  $f$ -free set.*

Hence,  $\text{FS}^r$  for  $r$ -trapped functions has strong cone avoidance property.

*Proof.* Fix  $A, X$  and  $f$  as in the assumption. We define a computable sequence of consecutive intervals as following. Let  $J_k = [a_k, b_k] = [k, k]$  for  $k < r$ . Given  $J_k = [a_k, b_k]$  defined and  $k+1 \geq r$ , let  $a_{k+1} = b_k + 1$ ,

$$b_{k+1} = \min\{b_k + 2^c : 2^c \geq 2^{k+3} \binom{k+1}{r}\}$$

and  $J_{k+1} = [a_{k+1}, b_{k+1}]$ . Let  $c_k$  be such that  $b_k - a_k = 2^{c_k} - 1$ .

Let  $T = \bigcup_{l < \omega} \prod_{k \leq l} J_k$ . Then  $T$  is a computably bounded computable subtree of  $[\omega]^{<\omega}$ . Moreover,  $[T]$  can be computably mapped to  $2^\omega$ : the string  $\sigma$  of length  $r$  such that  $\sigma(k) = k$  for all  $k < r$ , is mapped to the empty string; if  $\sigma \in T$  of length  $k > r$  is mapped to  $\mu \in 2^{<\omega}$  and  $x = a_k + i \leq b_k$ , then  $\sigma \langle x \rangle$  is mapped to  $\mu\nu$  where  $\nu$  is the  $i$ -th element of  $2^{c_k}$  under some computable enumeration of  $2^{<\omega}$ .

If  $\sigma \in T \cap [\omega]^k$  is  $f$ -free, then

$$\{x \in J_k : \sigma \langle x \rangle \text{ is not free for } f\} \subseteq \{f(\rho) : \rho \in [\sigma]^r\},$$

as  $f$  is  $r$ -trapped. So, for each  $l$ ,

$$m_{T \cap [\omega]^l} \{\sigma \in T \cap [\omega]^l : \sigma \text{ is free for } f\} > 2^{-1}.$$

Let  $S = \{\sigma \in T : \sigma \text{ is free for } f\}$ . Under the above computable isomorphism between  $[T]$  and  $2^\omega$ ,  $[S]$  is computably isomorphic to a  $\Pi_1^f$  class of Cantor space of positive measure. By the relativization of a result of Kučera (the corollary of Lemma 3 in [14], see also [17, Proposition 3.2.24]),  $X$  computes an infinite path  $Y \in [S]$  which is clearly free for  $f$ .

For the strong cone avoidance property, fix  $A \not\leq_T B$ . Then  $A \not\leq_T B \oplus X$  almost everywhere in Cantor space. So we can pick  $X$  and  $Y$  such that  $A \not\leq_T B \oplus X$ ,  $X$  is Martin-Löf random in  $f$ ,  $Y$  is an infinite  $f$ -free set computable in  $X$ .  $\square$

Now, we can finish (F1).

**Corollary 4.5.** *FS<sup>1</sup> has strong cone avoidance property.*

*Proof.* By Lemmata 4.3 and 4.4, we just need strong cone avoidance property of FS<sup>1</sup> for 0-trapped functions, which follows easily from Theorems 2.3 and 2.4.  $\square$

Assume that  $r > 1$  and FS <sup>$k$</sup>  for  $k < r$  has strong cone avoidance property. With these assumptions, we establish cone avoidance property of FS <sup>$r$</sup>  and thus accomplish (F2).

**Lemma 4.6.** *FS <sup>$r$</sup>  has cone avoidance property.*

*Proof.* Let  $X, Y$  and  $g : [\omega]^r \rightarrow \omega$  be such that  $X \not\leq_T Y \oplus g$ .

For each  $\sigma \in [\omega]^{r-1}$  and  $x$ , let  $R_{\sigma, x} = \{y > \max \sigma : g(\sigma \langle y \rangle) = x\}$ . By strong cone avoidance property of COH, let  $C \in [\omega]^\omega$  be such that  $C$  is cohesive for  $(R_{\sigma, x} : \sigma \in [\omega]^{r-1}, x < \omega)$  and  $X \not\leq_T Y \oplus g \oplus C$ . Thus, the following function is total:

$$\bar{g}(\sigma) = \begin{cases} \lim_{y \in C} g(\sigma \langle y \rangle), & \lim_{y \in C} g(\sigma \langle y \rangle) \text{ exists;} \\ \max \sigma, & \text{otherwise.} \end{cases}$$

By the induction hypothesis that FS <sup>$r-1$</sup>  has strong cone avoidance property, let  $D \in [C]^\omega$  be such that  $X \not\leq_T Y \oplus g \oplus C \oplus D$  and  $D$  is  $\bar{g}$ -free.

We define a desired  $g$ -free  $H$  as a subset of  $D$  by induction. Let  $\xi_0 = \emptyset$ . Suppose that  $\xi_s \in [D]^{<\omega}$  is defined and free for  $g$ . By the cohesiveness of  $C$ , if  $\sigma \in [\xi_s]^{r-1}$  and  $y \in C$  is sufficiently large, then either  $g(\sigma \langle y \rangle) = \lim_{y \in C} g(\sigma \langle y \rangle) = \bar{g}(\sigma)$ , or  $g(\sigma \langle y \rangle) > \max \xi_s$ . As  $\xi_s$  is a  $\bar{g}$ -free, if  $\sigma \in [\xi_s]^{r-1}$  and  $y \in C$  is sufficiently large, then either  $g(\sigma \langle y \rangle) = \bar{g}(\sigma) \notin \xi_s - \sigma = \xi_s \langle y \rangle - \sigma \langle y \rangle$ , or  $g(\sigma \langle y \rangle) > \max \xi_s$  and thus  $g(\sigma \langle y \rangle) \notin \xi_s \langle y \rangle - \sigma \langle y \rangle$  too. So the following number is defined:

$$x_s = \min\{y \in D : y > \max \xi_s \wedge \xi_s \langle y \rangle \text{ is free for } g\}.$$

Let  $\xi_{s+1} = \xi_s \langle x_s \rangle$ . Finally, let  $H = \bigcup_s \xi_s$ . Then  $H$  is  $g$ -free. Moreover,  $H \leq_T g \oplus D$  and thus  $X \not\leq_T Y \oplus g \oplus H$ .  $\square$

Below, we work on (F3): to prove strong cone avoidance property of FS <sup>$r$</sup> . By Lemmata 4.3 and 4.4, it suffices to prove the following restriction of Theorem 4.1.

**Lemma 4.7.** *For  $k < r$ , FS <sup>$r$</sup>  for  $k$ -trapped functions has strong cone avoidance property.*

From now on, we fix  $k < r$ ,  $A \not\leq_T B$  and a  $k$ -trapped function  $f : [\omega]^r \rightarrow \omega$ . If  $A \not\leq_T B \oplus X$ , then  $X$  is cone avoiding; a Mathias condition  $(\sigma, X)$  is cone avoiding if  $X$  is cone avoiding.

We prove Lemma 4.7 by constructing a cone avoiding infinite  $f$ -free set  $G$ . We build  $G$  in two steps:

- (1) We apply strong cone avoidance property of Achromatic Ramsey Theorems and of  $\text{FS}^q$  ( $q < r$ ) to build a cone avoiding  $E \in [\omega]^\omega$ , such that
  - (E) for each  $\rho \in [E]^{<\omega}$  with  $k < |\rho| < r$ , there exists  $b$  such that  $f(\rho\tau) \notin E - \rho$  for all  $\tau \in [E \cap (b, \infty)]^{r-|\rho|}$ .
- (2) By a Seetapun-style Mathias forcing, we build a cone avoiding  $f$ -free  $G \in [E]^\omega$ . In this step, we need some measure theoretic argument, which could be taken as an application of probabilistic method and is similar to that in Csima and Mileti [5]. The measure theoretic argument also needs strong cone avoidance property of Achromatic Ramsey Theorems.

To facilitate the construction, for each  $\sigma \in [\omega]^{<r}$ , let  $f_\sigma : [\omega]^{r-|\sigma|} \rightarrow \omega$  be such that  $f_\sigma(\tau) = f(\sigma\tau)$ . In particular,  $f_\emptyset = f$ . Moreover, fix  $(d_n : n > 0)$ , so that  $\text{ART}_{<\infty, d_n}^n$  has strong cone avoidance property.

**4.1. The construction of  $E$ .** We build a desired  $E$  from a cone avoiding  $D$ , which is sufficiently generic for Mathias forcing and has some nice properties.

**Lemma 4.8.** *For each  $\rho \in [\omega]^{<\omega}$  with  $k < |\rho| < r$  and a cone avoiding  $X \in [\omega]^\omega$ , there exist  $\theta \in [I_k^\rho]^{\leq d_{r-|\rho|}}$  and a cone avoiding  $Y \in [X]^\omega$  such that  $f_\rho([Y]^{r-|\rho|}) = \theta$ .*

*Proof.* As  $f$  is  $k$ -trapped and  $|\rho| > k$ ,  $f_\rho$  is a finite coloring with range contained by  $I_k^\rho$ . So the lemma follows from strong cone avoidance property of  $\text{ART}_{<\infty, d_{r-|\rho|}}^{r-|\rho|}$ .  $\square$

By the above lemma and Lemma 2.6, we can build a descending sequence of cone avoiding Mathias conditions  $((\sigma_n, X_n) : n < \omega)$  and a sequence of finite sets  $(\theta_\rho : k < |\rho| < r)$ , which satisfy the following properties:

- (1) for each  $n$ ,  $|\sigma_n| < |\sigma_{n+1}|$ ;
- (2) for each  $e$ , there exists  $n$  with  $\Phi_e^B(Z) \neq A$  for all  $Z \in (\sigma_n, X_n)$ ;

and also

- (E') if  $k < |\rho| < r$ , then  $\theta_\rho \in [I_k^\rho]^{\leq d_{r-|\rho|}}$  and  $f_\rho([X_n]^{r-|\rho|}) = \theta_\rho$  for some  $n$ .

Let  $D = \bigcup_n \sigma_n$ . Then  $D$  is infinite and cone avoiding.

For each  $l \in (k, r)$  and  $i < d_{r-l}$ , let  $F_{l,i} : [\omega]^l \rightarrow \omega$  be such that

$$F_{l,i}(\rho) = \begin{cases} \theta_\rho(i), & i < |\theta_\rho|; \\ 0, & \text{otherwise.} \end{cases}$$

By the induction hypothesis that  $\text{FS}^l$  for  $l < r$  has strong cone avoidance property, we can obtain a cone avoiding  $E \in [D]^\omega$ , which is  $F_{l,i}$ -free for all  $l \in (k, r)$  and  $i < d_{r-l}$ .

**Lemma 4.9.**  *$E$  satisfies (E).*

*Proof.* Fix an arbitrary  $\rho \in [E]^{<\omega}$  with  $l = |\rho| \in (k, r)$ . As  $E$  is  $F_{l,i}$ -free for all  $i < d_{r-l}$ ,  $\theta_\rho \cap (E - \rho) = \emptyset$ . By (E'), there exists  $b$  such that if  $\tau \in [E \cap (b, \infty)]^{r-l}$  then  $f_\rho(\tau) \in \theta_\rho$  and thus  $f(\rho\tau) = f_\rho(\tau) \notin E - \rho$ . So  $E$  satisfies (E).  $\square$

**4.2. The construction of  $G$ .** We build a desired  $f$ -free set  $G$  as a subset of  $E$ , by Mathias forcing.

We work with a specific subset of Mathias conditions. A Mathias condition  $(\sigma, X)$  is *admissible*, if  $\sigma X \subseteq E$ ,  $X$  is cone avoiding and  $\sigma\tau$  is  $f$ -free for all  $\tau \in [X]^{r-k}$ .  $(\emptyset, E)$  is trivially an admissible condition.

If  $(\sigma, X)$  is admissible, then let  $\mathcal{F}_{\sigma, X}$  be the set of all  $k$ -trapped  $g : [\omega]^r \rightarrow \omega$ , such that  $\sigma\tau$  is  $g$ -free for all  $\tau \in [X]^{r-k}$ . By the definition of admissibility,  $f \in \mathcal{F}_{\sigma, X}$ . As each  $k$ -trapped  $g$  satisfies  $g(\rho) \leq \max \rho$  for all  $\rho \in [\omega]^r$ ,  $\mathcal{F}_{\sigma, X}$  can be identified with a  $\Pi_1^X$  class in Cantor space.

By the lemma below, admissible conditions always capture some free sets.

**Lemma 4.10.** *If  $(\sigma, X)$  is an admissible Mathias condition and  $g \in \mathcal{F}_{\sigma, X}$ , then there exists  $Y \in [X]^\omega$  such that  $\sigma Y$  is  $g$ -free. Moreover, if  $g$  is cone avoiding then  $Y$  can also be cone avoiding.*

*Proof.* For each  $\rho \in [\sigma]^{<r}$ , let  $g_\rho$  be such that  $g_\rho(\tau) = g(\rho\tau)$  for all  $\tau \in [\omega]^{r-|\rho|}$  with  $\min \tau > \max \rho$ . By Free Set Theorems, pick  $Y \in [X]^\omega$  which is  $g_\rho$ -free for all  $\rho \in [\sigma]^{<r}$ ; if  $g$  is cone avoiding then  $Y$  can be also cone avoiding, by Lemma 4.6.

To show that  $\sigma Y$  is  $g$ -free, fix an arbitrary  $\xi \in [\sigma Y]^r$ . Let  $\rho = \xi \cap \sigma$  and  $\tau = \xi \cap Y$ .

We claim that  $g(\xi) \notin \sigma - \rho$ . If  $|\rho| < k$ , then  $g(\xi) \geq \xi(k-1) > \max \sigma$  as  $g$  is  $k$ -trapped. Suppose that  $|\rho| \geq k$ . Then  $\tau$  is contained by some  $\tau' \in [X]^{r-k}$  and  $\xi = \rho\tau \subseteq \sigma\tau'$ . As  $\sigma\tau'$  is  $g$ -free,  $g(\xi) \notin \sigma\tau' - \xi \supseteq \sigma - \rho$ .

On the other hand,  $g(\xi) \notin Y - \tau$  as  $Y$  is  $g_\rho$ -free and  $g(\xi) = g_\rho(\tau)$ .

So,  $\sigma Y$  is free for  $g$ .  $\square$

We introduce some conventions to facilitate a measure theoretic argument.

We fix  $d = d_{r-k}$  and  $c$  such that  $2^{c-1} > d$ .

A finite tree  $T \subset [\omega]^{<\omega}$  is *fast growing at order  $n$* , if for each  $\tau \in T - [T]$ ,

$$|\{x : \tau \langle x \rangle \in T\}| \geq 2^{|\tau|+c+2} \binom{n+|\tau|}{k}.$$

If  $(\sigma, X)$  is admissible and  $g \in \mathcal{F}_{\sigma, X}$ , then let  $\mathcal{T}(\sigma, X, g)$  be the set of all finite tree  $T \subset [X]^{<\omega}$ , such that  $T$  is fast growing at order  $|\sigma|$  and  $\sigma\tau$  is  $g$ -free for each  $\tau \in T$ .

According to the following two lemmata, it is promising to find finite sequences on a fast growing tree to extend an admissible condition.

**Lemma 4.11.** *Suppose that  $(\sigma, X)$  is admissible and  $T \in \mathcal{T}(\sigma, X, f)$ . Then there exists  $b$ , such that if  $\tau \in T$ ,  $\xi \in [E \cap (b, \infty)]^{r-k}$  and  $\sigma\tau\xi$  is  $f$ -free then*

$$|\{\tau \langle x \rangle \in T : \sigma\tau \langle x \rangle \xi \text{ is not free for } f\}| \leq \binom{|\sigma\tau|}{k}.$$

*Proof.* Let  $a < \omega$  be a strict upper bound of all numbers occurring in  $\sigma$  and  $T$ . By (E), pick  $b > a$ , such that for all  $\rho \subseteq a$  and  $v \subset E \cap (b, \infty)$ , if  $k < |\rho| < r$  and  $|\rho v| = r$  then  $f(\rho v) \notin E - \rho$ .

Fix  $\tau \in T$  and  $\xi \in [E \cap (b, \infty)]^{r-k}$  such that  $\sigma\tau\xi$  is  $f$ -free.

**Claim 4.12.** *If  $\tau \langle x \rangle \in T$  and  $\zeta = \rho \langle x \rangle v \in [\sigma\tau \langle x \rangle \xi]^r$ , then  $f(\zeta) \notin \sigma\tau \langle x \rangle \xi - \zeta$ .*

*Proof.* If  $v = \emptyset$ , then  $f(\zeta) = f(\rho \langle x \rangle) \leq \zeta(k) \leq x$  and  $f(\rho \langle x \rangle) \notin \sigma\tau \langle x \rangle - \rho \langle x \rangle$ , as  $f$  is  $k$ -trapped,  $k < r$  and  $\sigma\tau \langle x \rangle$  is  $f$ -free. So,  $f(\zeta) = f(\rho \langle x \rangle) \notin \sigma\tau \langle x \rangle \xi - \zeta$ .

If  $v \neq \emptyset$  and  $v \neq \xi$ , then  $|v| < r - k$  and  $|\rho \langle x \rangle| > k$ . As  $v \subset E \cap (b, \infty)$ ,  $f(\zeta) = f(\rho \langle x \rangle v) \notin E - \rho \langle x \rangle$ . As  $f$  is  $k$ -trapped,  $f(\zeta) \leq \zeta(k) \leq x$ . Thus  $f(\zeta) \notin E - \zeta \supset \sigma\tau \langle x \rangle \xi - \zeta$ .

If  $v = \xi$  then  $|\rho \langle x \rangle| = k$ . As  $f$  is  $k$ -trapped,  $f(\zeta) \geq \zeta(k-1) = x$ . So,  $f(\zeta) \notin \sigma\tau \supseteq \sigma\tau \langle x \rangle \xi - \zeta$ .  $\square$

By the above claim, if  $\tau \langle x \rangle \in T$  and  $\sigma\tau \langle x \rangle \xi$  is *not* free for  $f$ , then  $f(\zeta) \in \sigma\tau \langle x \rangle \xi - \zeta$  for some  $\zeta \in [\sigma\tau \xi]^r$ . As  $\sigma\tau \xi$  is  $f$ -free,  $f(\zeta) \notin \sigma\tau \xi - \zeta$ . Thus  $f(\zeta) = x$ . As  $f$  is  $k$ -trapped and  $\max \sigma\tau < x < \min \xi$ ,  $\zeta \cap \xi = \xi$ . Hence,

$$\{x : \tau \langle x \rangle \in T \wedge \sigma\tau \langle x \rangle \xi \text{ is not free for } f\} \subseteq \{f(\rho \xi) : \rho \in [\sigma\tau]^k\}.$$

The lemma follows immediately.  $\square$

**Lemma 4.13.** *Suppose that  $(\sigma, X)$  is admissible and  $T \in \mathcal{T}(\sigma, X, f)$ . Then there exists  $b$ , so that if  $\xi \in [E \cap (b, \infty)]^{r-k}$  and  $\sigma\xi$  is  $f$ -free then*

$$m_T\{\tau \in [T] : \sigma\tau\xi \text{ is not free for } f\} \leq 2^{-c-1}.$$

*Proof.* By the above lemma, for sufficiently large  $b$ , if  $\xi \in [E \cap (b, \infty)]^{r-k}$ ,  $\tau \in T - [T]$  and  $\sigma\tau\xi$  is  $f$ -free, then

$$m_T\{\tau\langle x \rangle \in T : \sigma\tau\langle x \rangle\xi \text{ is not free for } f\} \leq 2^{-|\tau|-c-2}m_T(\tau).$$

The lemma follows immediately from the above inequality.  $\square$

Now, we can extend an admissible condition to force a cone avoiding requirement.

**Lemma 4.14.** *For each  $e$  and an admissible  $(\sigma, X)$ , there exists an admissible  $(\tau, Y) \leq_M (\sigma, X)$  such that  $\Phi_e^B(Z) \neq A$  for all  $Z \in (\tau, Y)$ .*

*Proof.* Let  $\mathcal{U}$  be the set of  $g \in \mathcal{F}_{\sigma, X}$ , such that for every  $T \in \mathcal{T}(\sigma, X, g)$ ,

$$m_T\{v \in [T] : v \text{ contains an } (e, B)\text{-splitting pair over } \sigma\} < 2^{-1}.$$

So,  $\mathcal{U}$  is a  $\Pi_1^{B \oplus X}$  subset of  $\mathcal{F}_{\sigma, X}$ .

*Case 1:*  $\mathcal{U} = \emptyset$ . In particular,  $f \in \mathcal{F}_{\sigma, X} - \mathcal{U}$ .

Fix  $T \in \mathcal{T}(\sigma, X, f)$  such that

$$m_T\{v \in [T] : v \text{ contains an } (e, B)\text{-splitting pair over } \sigma\} \geq 2^{-1}.$$

Let  $b$  be as in Lemma 4.13 for  $(\sigma, X)$  and  $T$ , and let  $Y_0 = X \cap (b, \infty)$ .

For each  $\xi \in [Y_0]^{r-k}$ , let

$$h(\xi) = \{v \in [T] : \sigma v\xi \text{ is not free for } f\}.$$

So,  $h$  is a finite coloring of  $[Y_0]^{r-k}$ . By Lemma 4.13, each  $h(\xi)$  is a subset of  $[T]$  and  $m_T h(\xi) \leq 2^{-c-1}$ . By strong cone avoidance property of Achromatic Ramsey Theorems, there exist  $\theta$  and  $Y \in [Y_0]^\omega$ , such that  $|\theta| \leq d = d_{r-k}$ ,  $Y$  is cone avoiding and  $h(\xi) \in \theta$  for each  $\xi \in [Y]^{r-k}$ . Thus,

$$m_T\{v \in [T] : \exists S \in \theta(v \in S)\} \leq 2^{-c-1}d < 2^{-c-1}2^{c-1} = 2^{-2}.$$

Let  $P = \{v \in [T] : \forall S \in \theta(v \notin S)\}$ . By the definition of  $h$ ,

$$P = \{v \in [T] : \forall \xi \in [Y]^{r-k} (\sigma v\xi \text{ is free for } f)\};$$

by the above inequality,  $m_T P > 2^{-2}3$ .

So, we can pick  $v \in [T]$ , such that  $v$  contains an  $(e, B)$ -splitting pair over  $\sigma$  and  $\sigma v\xi$  is  $f$ -free for all  $\xi \in [Y]^{r-k}$ . Fix  $x$  and  $\eta \subseteq v$  such that  $\Phi_e^B(\eta; x) \not\downarrow A(x)$ . Let  $\tau = \sigma\eta$ . Then  $(\tau, Y)$  is a desired admissible condition.

*Case 2:*  $\mathcal{U} \neq \emptyset$ .

By Jockusch-Soare's Theorem 2.1, pick a cone avoiding  $g \in \mathcal{U}$ . By Lemma 4.10, let  $Y_0 \in [X]^\omega$  be such that  $Y_0$  is cone avoiding and  $\sigma Y_0$  is  $g$ -free. We define a  $Y_0$ -computable sequence of consecutive intervals by induction:

- $J_0 = [a_0, b_0]$ , where  $a_0 = 0$  and  $|Y_0 \cap J_0| = 2^{n_0}$  for some  $2^{n_0} \geq 2^{c+2} \binom{|\sigma|}{k}$ .
- if  $J_l = [a_l, b_l]$  is defined, then  $J_{l+1} = [a_{l+1}, b_{l+1}]$ , where  $a_{l+1} = b_l + 1$  and  $|Y_0 \cap J_{l+1}| = 2^{n_{l+1}}$  for some  $2^{n_{l+1}} \geq 2^{l+c+3} \binom{|\sigma|+l+1}{k}$ .

For each  $l$ , let  $T_l$  be the set of all  $v \in [\omega]^{\leq l}$  such that  $v(i) \in Y_0 \cap J_i$  for all  $i < |v|$ . Trivially,  $T_l \in \mathcal{T}(\sigma, X, g)$ . We can  $Y_0$ -computably map infinite paths of  $\bigcup_l T_l$  to  $2^\omega$ : the empty string is mapped to the empty string; if  $\sigma \in [T_l]$  is mapped to  $\mu \in 2^{<\omega}$  and  $x$  is the  $i$ -th element in  $Y_0 \cap J_l$ , then  $\sigma\langle x \rangle$  is mapped to  $\mu\nu$  such that  $\nu$  is the  $i$ -th element in  $2^{n_l}$  (under some computable enumeration of  $2^{<\omega}$ ).

As  $g \in \mathcal{U}$ , for each  $l$ ,

$$m_{T_l}\{v \in [T_l] : v \text{ contains an } (e, B)\text{-splitting pair over } \sigma\} < 2^{-1}.$$

Let

$$T = \{v \in \bigcup_l T_l : v \text{ contains no } (e, B)\text{-splitting pair over } \sigma\}.$$

Then under the above mapping,  $T$  is  $Y_0$ -computably isomorphic to a  $\Pi_1^{B \oplus Y_0}$  subset of Cantor space with positive measure. So, we can pick some  $R$ , such that  $R$  is Martin-Löf random in  $B \oplus Y_0$ ,  $A \not\leq_T B \oplus Y_0 \oplus R$  and  $Y_0 \oplus R$  computes some  $Y \in [T]$ . Then  $Y$  is cone avoiding and contains no  $(e, B)$ -splitting pair over  $\sigma$ . It follows that  $\Phi_e^B(Z) \neq A$  for all  $Z \in (\sigma, Y)$ . Hence,  $(\tau, Y)$  is as desired.  $\square$

By an argument similar to Case 1 in the proof of Lemma 4.14, we can extend the finite head of an admissible condition.

**Lemma 4.15.** *Each admissible condition  $(\sigma, X)$  admits an admissible extension  $(\tau, Y)$  with  $|\tau| > |\sigma|$ .*

*Proof.* Let  $n = 2^{c+2} \binom{|\sigma|}{k}$  and  $(x_i : i < n)$  be a strictly increasing sequence from  $X$ . Let  $T$  be a finite tree, consisting of exactly  $\emptyset$  and  $\langle x_i \rangle$  for  $i < n$ . Trivially,  $T \in \mathcal{T}(\sigma, X, f)$ . Let  $b$  be as in Lemma 4.11 for  $(\sigma, X)$  and  $T$ . For  $\xi \in [X \cap (b, \infty)]^{r-k}$ , let

$$h(\xi) = \{i : \sigma \langle x_i \rangle \xi \text{ is not free for } f\}.$$

By Lemma 4.11,  $h(\xi)$  is a subset of  $n$  with no more than  $\binom{|\sigma|}{k}$  elements. By strong cone avoidance property of ART, fix  $\theta$  and  $Y \in [X \cap (b, \infty)]^\omega$ , such that  $|\theta| \leq d$ ,  $Y$  is cone avoiding and  $h([Y]^{r-k}) = \theta$ . By the definition of  $h$ ,  $i \in S \in \theta$  if and only if  $\sigma \langle x_i \rangle \xi$  is not  $f$ -free for some  $\xi \in [Y]^{r-k}$ . So,

$$\{i < n : \forall S \in \theta (i \notin S)\} = \{i < n : \forall \xi \in [Y]^{r-k} (\sigma \langle x_i \rangle \xi \text{ is free for } f)\}.$$

Let  $N$  denote the set above. Then

$$|N| \geq n - |\theta| \binom{|\sigma|}{k} \geq (2^{c+2} - d) \binom{|\sigma|}{k} > 0.$$

So, we can pick  $i \in N$  and let  $\tau = \sigma \langle x_i \rangle$ . Then  $(\tau, Y)$  is as desired.  $\square$

With Lemmata 4.14 and 4.15, we can get a descending of admissible Mathias conditions  $((\sigma_n, X_n) : n < \omega)$  such that

- (1)  $(\sigma_0, X_0) = (\emptyset, E)$ ;
- (2)  $|\sigma_n| < |\sigma_{n+1}|$  for each  $n$ ;
- (3) for each  $n$  and  $Z \in (\sigma_{n+1}, X_{n+1})$ ,  $\Phi_n^B(Z) \neq A$ .

Let  $G = \bigcup_n \sigma_n$ . By admissibility,  $G$  is  $f$ -free; by the above properties,  $G$  is infinite and cone avoiding.

So, we prove Lemma 4.7 and thus also Theorem 4.1.

## 5. REMARKS AND QUESTIONS

As Jockusch's bounds apply for most theorems in the four families, if  $\Phi$  and  $\Psi$  are theorems from same family for exponents 2 and 3 respectively, then usually  $\Phi \not\vdash \Psi$ . Naturally, we expect to generalize this relation to larger exponents. In other words, we can ask whether any of the four families gives rise to a proper hierarchy of combinatorial principles below  $\text{ACA}_0$ . Actually, this question has been asked in [1, 5] for FS, TS and RRT respectively. In [20], it is shown that  $\text{RRT}_2^3 \not\vdash \text{RRT}_2^4$ . Here we state the parallel question for ART.

**Question 5.1.** *Fix  $(d_k : 0 < k < \omega)$  as in §3. Does  $\text{ART}_{c,d_r}^r \vdash \text{ART}_{e,d_{r+1}}^{r+1}$  for any  $r > 1$  and reasonable  $c, e$ ?*

A possible approach to answer the above questions would be to construct relating solutions with humble iterated jumps, as the author did in [20]. Recall that a set  $X$  is  $\text{low}_n$  if  $X^{(n)} \equiv_T \emptyset^{(n)}$ ; otherwise,  $X$  is *non-low<sub>n</sub>*. By Cholak, Jockusch and Slaman [2],  $\text{RT}_2^2$  admits non-low<sub>2</sub>-omitting, and so do all  $\Phi$  in the four families for exponent 2, as they are consequences of  $\text{RT}_2^2$ ; by [20],  $\text{RRT}_2^3$  admits non-low<sub>3</sub>-omitting. But in general, we do not know much.

**Question 5.2.** *Given  $\Phi$  which is a statement in the four families with exponent  $r > 2$ , does it admit non-low $_r$ -omitting?*

For ART, we can ask finer questions. Clearly, for fixed  $r$  and  $c$ , if  $d \leq e$  then  $\text{ART}_{c,d}^r \vdash \text{ART}_{c,e}^r$ . Dorais et al. [6] have established some implications between different instances of ART, e.g.,  $\text{ART}_{k^n, k^n-1}^{mn+1} \vdash \text{ART}_{k, k-1}^{m+1}$  ([6, Proposition 5.3]), although they use a less artistic name for some instances of ART here. But in general, relations between distinct instances of ART are unknown.

**Question 5.3.** *Compare distinct instances of ART, e.g.,  $\text{ART}_{c,d}^r$  and  $\text{ART}_{c,d+1}^r$ .*

Note that, if  $c < \infty$  then  $\text{ART}_{c,d}^r$  is equivalent to  $\text{ART}_{d+1,d}^r$ . However, the obvious proof for this equivalence can not be generalized to yield  $\text{ART}_{d+1,d}^r \vdash \text{ART}_{<\infty,d}^r$ , even if  $d$  is a standard positive integer.

**Question 5.4.** *Compare  $\text{ART}_{d+1,d}^r$  and  $\text{ART}_{<\infty,d}^r$ .*

Another kind of metamathematical questions is about the relations between the four families. Recently, Xiaojun Kang [13] proves that  $\text{RRT}_2^2 \not\vdash \text{TS}^2$  and thus  $\text{RRT}_2^2$  is strictly weaker than  $\text{FS}^2$ . The general picture is yet to be discovered.

**Question 5.5.** *Compare theorems between different families.*

People may also be interested in the integer series  $(d_k : 0 < k < \omega)$  in §3. By the proof of Theorem 3.1, we can take the series (with offset 1) to be the Schröder numbers (see [22]):

$$S_0 = 1, \quad S_n = S_{n-1} + \sum_{k < n} S_k S_{n-k-1}.$$

**Question 5.6.** *Are Schröder numbers optimal bounds for Achromatic Ramsey Theorems to have strong cone avoidance property? Or just for  $\text{ART}_{<\infty,d}^r \not\vdash \text{ACA}_0$ ?*

Dorais et al. [6, Proposition 5.5] have shown that  $\text{ART}_{2^r, 2^r-1}^{r+1} \vdash \text{ACA}_0$ . But the gap between  $2^r - 1$  and  $S_r$  is quite large, as  $S_r > 2^{2^r-2}$ .

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